

THE ZERO-DIVISOR GRAPHS OF RINGS AND SEMIRINGS

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ABSTRACT. In this paper we study zero-divisor graphs of rings and semirings. We show that all zero-divisor graphs of (possibly noncommutative) semirings are connected and have diameter less than or equal to 3. We characterize all acyclic zero-divisor graphs of semirings and prove that in the case zero-divisor graphs are cyclic, their girths are less than or equal to 4. We find all possible cyclic zero-divisor graphs over commutative semirings having at most one 3-cycle, and characterize all complete k -partite and regular zero-divisor graphs. Moreover, we characterize all additively cancellative commutative semirings and all commutative rings such that their zero-divisor graph has exactly one 3-cycle.

1. INTRODUCTION

For any semigroup S with zero, we denote by $Z(S)$ the set of zero-divisors, $Z(S) = \{x \in S; \text{there exists } 0 \neq y \in S \text{ such that } xy = 0 \text{ or } yx = 0\}$. We denote by $\Gamma(S)$ the *zero-divisor graph* of S . The vertex set $V(\Gamma(S))$ of $\Gamma(S)$ is the set of elements in $Z(S)^* = Z(S) \setminus \{0\}$ and an unordered pair of vertices $x, y \in V(\Gamma(S))$, $x \neq y$, is an edge $x - y$ in $\Gamma(S)$ if $xy = 0$ or $yx = 0$.

Similarly, we can define the zero-divisor graphs of other algebraic structures, e.g. rings, semirings, near-rings, algebras.

The zero-divisor graphs of rings have been first introduced by Beck in [11] in the study of commutative rings, and later studied by various authors, see for example [1, 2, 3, 7, 5, 8, 12, 14, 21, 24]. The zero-divisor graphs are also intensely studied in the semigroup setting, e.g. [16, 17, 18, 19]. Recently, they were used to study near-rings (see e.g. [13]) and semirings (see e.g. [9, 10]).

In this paper we investigate the interplay between the algebraic properties of a (semi)ring and the graph theoretic properties of its zero-divisor graph. In the next section, we give all necessary definitions. In Section 3, we survey some of the known results of the theory of the zero-divisor graphs over semigroups, rings, and semirings, and extend these results to a more general setting of a noncommutative semiring and we characterize all acyclic zero-divisor graphs of semirings (Theorem 3.9). Next, we study the cyclic zero-divisor graphs. Firstly, we characterize the complete k -partite and regular zero-divisor graphs that can appear as the zero-divisor graphs of commutative semirings (Theorem 4.2 and Corollary 4.3). In the case the zero-divisor graph of a commutative semiring contains at most one triangle, we find all possible zero-divisor graphs (Theorems 6.4 and 7.5, Proposition 7.1). If the zero-divisor graph of a commutative semiring is cyclic and contains no triangles,

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we describe the order of the nilpotent elements in the semiring (Proposition 6.5). In the case the zero-divisor graph of an additively cancellative semiring contains exactly one triangle, we prove that the semiring has to be a ring (Proposition 7.7) and we then proceed to characterize all rings and their zero-divisor graphs containing exactly one triangle (Theorem 8.4).

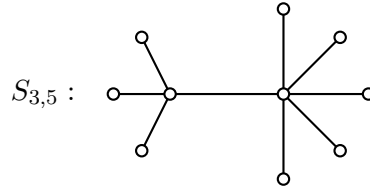
2. DEFINITIONS

A *semiring* is a set S equipped with binary operations $+$ and \cdot such that $(S, +)$ is a commutative monoid with identity element 0, and (S, \cdot) is a monoid with identity element 1. In addition, operations $+$ and \cdot are connected by distributivity and 0 annihilates S . A semiring is *commutative* if $ab = ba$ for all $a, b \in S$. A semiring is *entire* (or *zero-divisor-free*) if $ab = 0$ implies that $a = 0$ or $b = 0$. The semiring S is *additively cancellative* if $a + c = b + c$ implies that $a = b$ for all $a, b, c \in S$.

The simplest example of a commutative semiring is the *binary Boolean semiring*, the set $\{0, 1\}$ in which $1 + 1 = 1 \cdot 1 = 1$. We denote the binary Boolean semiring by \mathcal{B} . Moreover, the set of nonnegative integers (or reals) with the usual operations of addition and multiplication, is a commutative semiring. Other examples of commutative semirings are distributive lattices, tropical semirings etc.

The sequence of edges $x_0 - x_1, x_1 - x_2, \dots, x_{k-1} - x_k$ in a graph is called a *path of length k* and is denoted by $x_0 - x_1 - \dots - x_k$ or P_{k+1} . The *distance* between two vertices is the length of the shortest path between them. The *diameter* $\text{diam}(\Gamma)$ of the graph Γ is the longest distance between any two vertices of the graph. A path $x_0 - x_1 - \dots - x_{k-1} - x_0$ is called a *cycle*. The *girth* of the graph Γ is the length of the shortest cycle contained in the graph and will be denoted by $\text{girth}(\Gamma)$.

The complete graph will be denoted by K_n and complete bipartite graph by $K_{m,n}$. We say that the *star graph* is a complete bipartite graph $K_{1,n}$. Note that $K_{1,0} = K_1$. The *two-star graph* $S_{m,n}$, where $n, m \in \mathbb{N} \cup \{0\}$, is a graph with the set of vertices equal to the set $\{v_1, v_2, u_1, u_2, \dots, u_m, w_1, w_2, \dots, w_n\}$, and with the following edges: $v_1 - v_2, u_i - v_1$ for $i = 1, 2, \dots, m$, and $w_j - v_2$ for $j = 1, 2, \dots, n$. Note that $S_{0,n} = K_{1,n+1}$ is a star graph.



Let $\overline{K}_{m,n}^r$ be the complete bipartite graph $K_{m,n}$ together with r vertices v_1, v_2, \dots, v_r and edges $v_i - a$ for all i and some vertex a , such that $\deg(a) = n$ in the induced subgraph $K_{m,n}$. Moreover, choose vertex b , such that $\deg(b) = m$ and $a - b$ is an edge in $\overline{K}_{m,n}^r$. Denote by $K_{m,n}^{\triangle(r_1, r_2, r_3)}$ the graph $\overline{K}_{m,n}^{r_1}$ together with vertices e, v_i, w_j and edges $a - e - b, b - v_i, e - w_j, i = 1, 2, \dots, r_2, j = 1, 2, \dots, r_3$.



3. THE ZERO-DIVISOR GRAPH OF A SEMIRING

Let us investigate the zero-divisor graph of an arbitrary (possibly noncommutative) semiring.

Firstly, we shall prove that the zero-divisor graphs of (noncommutative) semirings are always connected and have diameters at most 3. This is a similar result to [24, Thm. 3.2] (for rings) and [9, Lemma 2.1] (for commutative semirings).

Theorem 3.1. *If S is a semiring, then $\text{diam}(\Gamma(S)) \leq 3$.*

Proof. Take $x, y \in Z(S)^*$, such that $xy \neq 0$ and $yx \neq 0$. We want to show that there is a path from x to y and $d(x, y) \leq 3$.

There exist $a, b \in Z(S)^*$, such that $ax = 0$ or $xa = 0$ and $by = 0$ or $yb = 0$. Note that here, a can be equal to x , as well as b equal to y . If $a = b$, $ab = 0$, $ba = 0$, $ay = 0$, $ya = 0$, $bx = 0$ or $xb = 0$, then $d(x, y) \leq 3$. So, suppose, none of the above is true. In the case $ax = 0$, we have that either $x - ba - y$ or $x - ya - b - y$ is a path joining x and y . Otherwise, if $xa = 0$, either $x - ab - y$ or $x - ay - b - y$ is a path from x to y . All of these four paths are of length at most 3, even if some of the vertices coincide, and therefore $d(x, y) \leq 3$. ■

Anderson and Mulay [8, Thm. 2.8] proved that for direct products of integral domains and their subrings, the diameter is at most 2. We generalize this result to noncommutative entire semirings.

Proposition 3.2. *If S_1 and S_2 are entire semirings and $S \subseteq S_1 \times S_2$ is a semiring. If $\Gamma(S) \neq \emptyset$, then $\text{diam}(\Gamma(S)) \leq 2$.*

Proof. If $S \subseteq S_1 \times S_2$, where S_1 and S_2 are entire semirings, then $(s_1, s_2) \in Z(S)^*$ implies that either $s_1 = 0$ or $s_2 = 0$.

Assume $\text{diam}(\Gamma(S)) \geq 2$. Then, there exist $x, y \in Z(S)^*$, such that $xy \neq 0$, $yx \neq 0$. Without loss of generality, let us assume that $x = (x_1, 0)$. This implies that $y = (y_1, 0)$. Since $\Gamma(S)$ is a connected graph, there exists an edge $x - z$ in $\Gamma(S)$ and $z = (0, z_1)$. Thus $x - y - z$ is a path in $\Gamma(S)$ and $\text{diam}(\Gamma(S)) \leq 2$. ■

In the following examples we show that some families of the graphs with $\text{diam}(\Gamma) \leq 2$ can be realized as the zero-divisor graphs of semirings. We will later need the realization of these families of graphs in the characterization of complete k -partite and regular zero-divisor graphs.

We will denote by $\mathcal{M}_n(S)$ the set of all $n \times n$ matrices over a semiring S . The matrix with the only nonzero entry 1 in the i th row and j th column will be denoted by $E_{i,j}$. The matrix I_n will denote the $n \times n$ identity matrix, 0_n will denote the $n \times n$ zero matrix and J_n will denote the matrix $E_{1,2} + E_{2,3} + \dots + E_{n-1,n}$. Also, let us denote by $A \oplus B$ the direct sum of matrix blocks $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$.

Example 3.3. *We can realize all complete bipartite graphs as zero-divisor graphs of a direct product of two semirings. Namely, if S and T are arbitrary entire semirings with $|S| = m + 1$ and $|T| = n + 1$, then $\Gamma(S \times T) = K_{n,m}$. Such S and T exist, for example we can choose totally ordered sets of appropriate cardinality.* \square

Example 3.4. *Anderson and Livingston [7, Thm. 2.8] proved that if the zero-divisor graph of a commutative ring R with 1 is equal to $\Gamma(R) = K_n$, $n \geq 3$, then all zero-divisors are nilpotents of order 2. This statement does not hold for semirings. However, we can show that if S is a commutative semiring and $\Gamma(S) = K_n$, $n \geq 3$, then $x^2 = 0$ for all but possibly one $x \in Z(S)^*$.*

Suppose $Z(S)^ = \{a_1, a_2, \dots, a_n\}$ and $a_1^2 \neq 0$. Since $\Gamma(S) = K_n$, it follows that $a_i a_j = 0$ for all $i \neq j$, and therefore $a_1(a_1 + a_i) = a_1^2 \neq 0$ for $i \neq 1$. Since $a_j(a_1 + a_i) = 0$ for all $j \neq i$, $j, i \neq 1$, it follows that $a_1 + a_i \in Z(S)^*$, thus $a_1 + a_i = a_1$ for all $i \neq 1$. By multiplying this equation by a_i , we have $a_i^2 = 0$ for all $i \neq 1$.*

Moreover, such semirings S indeed exist. Consider a semiring S in $\mathcal{M}_{2n-1}(\mathcal{B})$, generated by $a_i = J^{n-2+i} + J^{n-1+i} + \dots + J^{2n-2}$, where $i = 1, 2, \dots, n$. Since $J^{2n-1} = 0$, we have $a_i a_j = 0$ for all $i, j = 1, 2, \dots, n$ but for $i = j = 1$. Therefore $\Gamma(S) = K_n$, $a_i^2 = 0$ for all $i \neq 1$ and $a_1^2 \neq 0$. \square

The next two examples show that we can also realize all possible star and two-star graphs as the zero-divisor graphs of (even commutative) semirings. Compare this with [7, Ex. 2.1] where it has been shown that for a commutative ring, the zero-divisor graph cannot be equal to $P_4 = S_{1,1}$.

Example 3.5. *Let $\mathcal{M}_{n+1}(\mathcal{B})$ be the semiring of $n + 1$ by $n + 1$ matrices over the Boolean semiring, where $n \geq 1$, and denote by S the subsemiring generated by the set $\{I_1 \oplus 0_n, 0_1 \oplus I_n, 0_1 \oplus I_n + J_n\}$. The zero-divisors in the semiring S are of two types, $I_1 \oplus 0_n$ and $0_1 \oplus (I_n + J_n + J_n^2 + \dots + J_n^k)$ for $k = 0, 1, \dots, n - 1$. It can be easily verified that then only the products of the element $I_1 \oplus 0_n$ with $0_1 \oplus (I_n + J_n + J_n^2 + \dots + J_n^k)$ are equal to zero for all k , so $\Gamma(S) = K_{1,n} = S_{0,n-1}$.*

Obviously, we can realize the graph $K_1 = K_{1,0}$ as the zero-divisor graph of a semiring, for example the (semi)ring \mathbb{Z}_4 . \square

Example 3.6. *Choose $n, m \in \mathbb{N}$. Let $L = \{0, x_1, x_2, x_3, \dots, 1\}$ be any totally ordered (distributive) lattice containing at least $\max\{n, m\}$ nonzero elements. Then L is also an entire semiring for the operations $x_i + x_j = x_{\max\{i,j\}}$ and $x_i \cdot x_j = x_i x_j = x_{\min\{i,j\}}$.*

Now, let $\mathcal{M}_4(L)$ denote the semiring of all 4×4 matrices over L . Denote $v_1 = 0_2 \oplus x_1 J_2$ and $v_2 = (x_1(I_2 + J_2)) \oplus 0_2$. For $i = 1, 2, \dots, n$ define $u_i = (x_1 I_2 + x_i J_2) \oplus x_1 J_2$, and for

$j = 1, 2, \dots, m$ define $w_j = 0_2 \oplus (x_j(I_2 + J_2))$. Let S denote the subsemiring of $\mathcal{M}_4(L)$, generated by the elements $v_1, v_2, u_1, u_2, \dots, u_n, w_1, w_2, \dots, w_m$.

It can be easily seen, that since L is entire and antinegative, we do not get any zero-divisors in S that are not already amongst the generating elements. So, the zero-divisor graph of S consists of edges v_1-v_2 , u_i-v_1 for $i = 1, 2, \dots, n$, and w_j-v_2 for $j = 1, 2, \dots, m$, which implies that $\Gamma(S) = S_{n,m}$ is a two-star graph. \square

We shall now see, that we can consider the case of cyclic zero-divisor graphs separately from the case of acyclic ones. We will find all possible acyclic graphs that can be realized as zero-divisor graphs of semirings, and for the cyclic graphs, we shall prove that they always contain at least one cycle of length at most 4.

Lemma 3.7. *If P_5 is a subgraph of $\Gamma(S)$, where S is an arbitrary semiring, then $\Gamma(S)$ is a cyclic graph and $\text{girth}(\Gamma(S)) \leq 4$.*

Proof. Denote by $a-b-c-d-e$ the path P_5 in $\Gamma(S)$. Suppose that $\text{girth}(\Gamma(S)) > 4$, i.e., there are no edges among other vertices from this path.

Consider first the case $ba = bc = 0$. Since $eb \neq 0$ and $(eb)a = (eb)c = 0$, we have that $eb = b$. (Otherwise, there is a cycle of length 3 or 4 in $\Gamma(S)$.) Similarly, we conclude that $db = b$. Since $d-e$ is an edge in $\Gamma(S)$, we have either that $de = 0$, and thus $db = dbe = 0$, or $ed = 0$, and therefore $eb = edb = 0$, which both contradict the assumption that $\text{girth}(\Gamma(S)) > 4$.

Similarly, we can treat the case $ab = cb = 0$.

Suppose now $ab = bc = 0$ and $ba \neq 0$, $cb \neq 0$. By Theorem 3.1, we have that $d(a, e) \leq 3$. Since $\text{girth}(\Gamma(S)) > 4$, the path from a to e of the length at most 3 cannot contain any of vertices b, c, d . If $d(a, e) = 3$ and $a-x-y-e$ is a path from a to e , we obtain a cycle $a-b-c-d-e-y-x-a$ of length 7. Note that if $d(a, e) = 2$, then we can assume that $y = x$ and if $d(a, e) = 1$, then $e = y = x$. In all three cases, let us assume, that $a-x$ is an edge in $\Gamma(S)$. If we assumed $ax = 0$, we would get a contradiction as in the case $ba = bc = 0$. Thus, from now, let $xa = 0$. Since $\text{girth}(\Gamma(S)) > 4$ and $b(cx) = 0$, the product cx is either equal to a, b, c or is an element, different from a, b, c, d, e, f, x, y . In the first case, $a^2 = (cx)a = 0$ and therefore $a-ac-x-a$ is a 3-cycle in $\Gamma(S)$, a contradiction. In the second case, $ba = (cx)a = 0$, which is again a contradiction. Otherwise, $b-cx-a-b$ is a cycle of length 3. \blacksquare

Corollary 3.8. *The cycle on n vertices, $n \geq 5$, cannot be realized as $\Gamma(S)$, where S is a semiring.*

We can now prove the theorem that generalizes [8, Thm. 2.4, Thm. 2.5] and provides a characterization of all acyclic zero-divisor graphs over semirings.

Theorem 3.9. *Let S be a non-entire semiring.*

- (a.) *If $\Gamma(S)$ is a cyclic graph, then $\text{diam}(\Gamma(S)) \leq 3$ and $\text{girth}(\Gamma(S)) \leq 4$.*
- (b.) *Γ is an acyclic zero-divisor graph of a non-entire semiring if and only if $\Gamma = S_{n,m}$ or $\Gamma = K_1$.*

Proof. If $\Gamma(S)$ is a cyclic graph which contains a cycle of length 5 or more, then it also contains P_5 . By Lemma 3.7, it follows $\text{girth}(\Gamma(S)) \leq 4$. Note that $\text{diam}(\Gamma(S)) \leq 3$ by Theorem 3.1. Assume now that $\Gamma(S)$ is acyclic and contains at least 2 vertices. Again by Lemma 3.7, we know that it does not contain P_5 , so the only possibility is that $\Gamma(S) = S_{n,m}$ for some n, m . The converse of (b.) follows from Examples 3.5 and 3.6. ■

This result characterizes the acyclic zero-divisor graphs of (non)-commutative semirings. In the following sections we will examine the cyclic zero-divisor graphs of commutative semirings.

4. THE COMPLETE k -PARTITE AND REGULAR ZERO-DIVISOR GRAPHS OF COMMUTATIVE SEMIRINGS

In this section we investigate two special families of cyclic graphs, complete k -partite and regular graphs. DeMeyer et al. [18] showed that all complete k -partite graphs are zero-divisor graphs of commutative semigroups, and (see Theorem 4.1) characterized all regular graphs that can appear as the zero-divisor graphs of commutative semigroups. In the semiring setting, these two assertions no longer hold, and in Theorem 4.2 and Corollary 4.3 we shall characterize complete k -partite and regular graphs that can appear as the zero-divisor graphs of commutative semirings.

Theorem 4.1 (DeMeyer, Greve, Sabbaghi, Wang [18]). *Let Γ be a connected k -regular graph with n vertices. Then Γ is a zero-divisor graph of a commutative semigroup if and only if $n - k \mid n$ and $\Gamma = \bigvee^{n/(n-k)} (n - k) K_1$.*

Theorem 4.2. *Let Γ be a complete k -partite graph with n vertices and $k \geq 2$. Then Γ is a zero-divisor graph of a commutative semiring if and only if $k = 2$ or $\Gamma = K_{k-1} \bigvee (n - k + 1) K_1$.*

Proof. Since Γ is connected, we have $k \geq 2$. Suppose $\Gamma = C_1 \bigvee C_2 \bigvee \dots \bigvee C_k$ is a complete k -partite zero-divisor graph with $k \geq 3$. If $\Gamma \neq K_{k-1} \bigvee (n - k + 1) K_1$, then there exist, say C_1 and C_2 with $|C_1| \geq 2$ and $|C_2| \geq 2$. Let $a_1, b_1 \in C_1$, $a_2, b_2 \in C_2$ and $c \in C_3$. Since $a_1 c = a_2 c = 0$, it follows that $(a_1 + a_2) c = 0$, so $a_1 + a_2 \in Z(S)$. If $a_1 + a_2 \notin C_1$, then $(a_1 + a_2) b_1 = 0$ and thus $a_1 b_1 = 0$, a contradiction. Therefore, $a_1 + a_2 \in C_1$ and similarly we obtain $a_1 + a_2 \in C_2$ which is also a contradiction.

Example 3.3 shows that $K_{m,n}$ can be realized as the zero-divisor graph of a commutative semiring. Choose an integer k , $3 \leq k \leq n - 1$ and consider the subsemiring $S \subseteq \mathcal{M}_{2n+1}(\mathcal{B})$, generated by matrices $\{A_i, B, C_j; 2 \leq i \leq n - k + 1, 2n - k + 2 \leq j \leq 2n\}$, where $J = J_{2n+1}$, $B = J^n + J^{n+1} + \dots + J^{2n}$, $C_j = J^j + J^{j+1} + \dots + J^{2n}$ and $A_i = [a_{st}^i]$ are the matrices with entries

$$a_{st}^i = \begin{cases} 0, & t - s < n \text{ or } s = i, t = i + n, \\ 1, & \text{otherwise.} \end{cases}$$

Observe that S is a semiring with $\Gamma(S) = \{C_1, C_2, \dots, C_k\} \bigvee \{A_1, A_2, \dots, A_{n-k+1}, B\} = K_{k-1} \bigvee (n - k + 1) K_1$. ■

Corollary 4.3. *Let Γ be a r -regular graph with n vertices. Then Γ is a zero-divisor graph of a commutative semiring if and only if $r = n - 1$ and $\Gamma = K_n$, or n even, $r = \frac{n}{2}$ and $\Gamma = K_{\frac{n}{2}, \frac{n}{2}}$.*

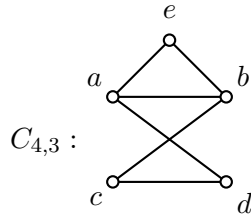
Proof. Assume that Γ is a zero-divisor graph of a semiring. Then, Γ is connected by Theorem 3.1 and thus by 4.1, Γ is a join of $\frac{n}{n-r}$ copies of $(n-r)K_1$, which is a complete $\frac{n}{n-r}$ -partite graph. Now, Theorem 4.2 implies that there are two possibilities. In the first case, $\frac{n}{n-r} = 2$ and thus Γ is a r -regular bipartite graph with $r = \frac{n}{2}$, so $\Gamma = K_{\frac{n}{2}, \frac{n}{2}}$. In the second case, $\Gamma = K_{\frac{r}{n-r}} \vee (n - \frac{r}{n-r})K_1$. Since Γ is r -regular, it follows that $n - 1 = \frac{r}{n-r}$ and therefore $r = n - 1$, so $\Gamma = K_n$.

Examples 3.3 and 3.4 show that K_n and $K_{\frac{n}{2}, \frac{n}{2}}$ can both be realized as the zero-divisor graphs of commutative semirings. ■

5. THE CYCLIC ZERO-DIVISOR GRAPHS

In this section we will study the cyclic zero-divisor graphs of commutative semirings. By Theorem 3.9 every cyclic zero-divisor graph has a 3-cycle or a 4-cycle. We will define the following graphs, which we shall prove are the graphs that cannot appear as the induced subgraphs of a cyclic zero-divisor graph of a commutative semiring.

- $C_{4,4}$, which is a graph consisting of two cycles $a - b - c - d - a$ and $a - b - f - e - a$ with the common edge $a - b$, and
- C'_4 , which is a 4-cycle $a - b - c - d - a$ together with two vertices e and f , connected with edges $a - e$ and $b - f$,
- C''_4 , which is a 4-cycle $a - b - c - d - a$ together with two vertices e and f , connected with edges $a - e$ and $c - f$,



Moreover, let us define the graph $C_{4,3}$, which is a graph consisting of a 4-cycle $a - b - c - d - a$ and a 3-cycle $a - b - e - a$ with the common edge $a - b$.

We can now state the following lemma.

Lemma 5.1. *Let S be a commutative semiring and let the graph $\Gamma(S)$ contain exactly one 3-cycle and at least one n -cycle, $n \geq 4$. Then, $\Gamma(S)$ contains $C_{4,3}$ as an induced subgraph.*

Proof. Let $n \geq 4$ be the smallest integer, such that $\Gamma(S)$ contains an n -cycle $x_1 - x_2 - x_3 - \dots - x_n - x_1$. If $x_1 x_3 = 0$, we obtain a $n - 1$ cycle in the graph $\Gamma(S)$ and thus $n = 4$ and Γ has two 3-cycles, a contradiction. Since $(x_1 x_3)x_2 = (x_1 x_3)x_4 = (x_1 x_3)x_n = 0$, either $x_1 x_3$ is a vertex on the cycle or $x_1 x_3 \neq x_i$ for all i . In both cases $\Gamma(S)$ contains a 4-cycle.

Suppose $\Gamma(S)$ contains a 4-cycle $a - b - c - d - a$ and a 3-cycle $e - f - g - e$. We shall firstly prove that they have a common vertex.

Choose an arbitrary vertex in the 3-cycle, say e . If e is a neighbour of at least 2 vertices from the 4-cycle (say, one of them is a), then note that either $\Gamma(S)$ contains more than one 3-cycle (which contradicts the assumption), or the only other neighbour of e in the 4-cycle is c . In the latter case $\Gamma(S)$ contains a 4-cycle (either $a - e - c - b - a$ or $a - e - c - d - a$) and the 3-cycle $e - f - g - e$ with the common vertex e . So, suppose every vertex in the 3-cycle $e - f - g - e$ has at most one neighbour in the 4-cycle. In this case, there exists a vertex in the 4-cycle, say a , such that $ae \neq 0$. Since $(ae)f = (ae)g = 0$ and $\Gamma(S)$ has only one 3-cycle, it follows that ae is an element of $\{e, f, g\}$. On the other hand, $(ae)b = (ae)d = 0$, so ae has at least 2 neighbours in the 4-cycle. It follows that $ae = c$ and 4-cycle and 3-cycle have a common vertex.

We proved that 3-cycle and 4-cycle have a common vertex, for instance $d = g$. If $a = e$, then the Lemma is proven. Otherwise, since the graph contains only one 3-cycle, $ae \neq 0$ and $(ae)d = (ae)f = 0$ implies that ae is an element of $\{d, e, f\}$. Moreover, $(ae)b = 0$ and thus $\Gamma(S)$ contains $C_{4,3}$, since $\Gamma(S)$ may contain only one 3-cycle. ■

Lemma 5.2. *Let S be a commutative semiring with $\text{girth}(\Gamma(S)) = 4$. Then C'_4 cannot appear as an induced subgraph of $\Gamma(S)$.*

Proof. Suppose that $\Gamma(S)$ contains C'_4 , which is a 4-cycle $a - b - c - d - a$ together with two vertices e and f , connected with edges $a - e$ and $b - f$. Firstly, $ef \neq 0$ and $(ef)a = (ef)b = 0$. Again, since $\text{girth}(\Gamma(S)) > 3$, it follows that either $ef = a$ or $ef = b$. By the symmetry, we can assume that $ef = a$. Now, $e(fd) = (ef)d = 0$ and moreover $a(fd) = b(fd) = c(fd) = 0$. Since $fd \neq 0$ and $\text{girth}(\Gamma(S)) > 3$, the product fd cannot exist as a vertex in $\Gamma(S)$. ■

Lemma 5.3. *Let S be a commutative semiring with $\Gamma(S)$ containing at most one 3-cycle. Then neither C''_4 , $C_{4,4}$, nor $C_{4,5}$ can appear as induced subgraphs of $\Gamma(S)$.*

Proof. Suppose $\Gamma(S)$ contains $C_{4,4}$ as an induced subgraph, i.e. $\Gamma(S)$ contains vertices a, b, c, d, e, f , where the only edges are $a - b - c - d - a$ and $a - b - f - e - a$. Consider the product ce . Clearly, $(ce)a = (ce)b = (ce)d = (ce)f = 0$ and $ce \neq 0$, and since $\Gamma(S)$ contains at most one 3-cycle, such vertex ce cannot exist in $\Gamma(S)$.

If $\Gamma(S)$ contains $C_{4,5}$ as an induced subgraph, i.e. $\Gamma(S)$ contains vertices a, b, c, d, e, f , where the only edges are $a - b - c - d - a$ and $a - b - c - f - e - a$, then similarly as in (1), $ce \neq 0$ is a zero divisor in S , but cannot exist as a vertex in $\Gamma(S)$.

If $\Gamma(S)$ contains C''_4 , which is a 4-cycle $a - b - c - d - a$ together with two vertices e and f , connected with edges $a - e$ and $c - f$, note that $ce \neq 0$ and $(ce)a = (ce)b = (ce)d = (ce)f = 0$. Since $\Gamma(S)$ contains at most one 3-cycle, it follows that ce cannot exist as a vertex in $\Gamma(S)$. ■

6. COMMUTATIVE SEMIRINGS WITH ZERO-DIVISOR GRAPHS OF GIRTH 4

In this section we shall describe the zero-divisor graphs of commutative semirings with their girth equal to 4.

If the semiring is a ring, the structure of the ring itself can be deduced from the properties of its zero-divisor graph. Anderson and Mulay, [8, Theorems 2.3 and 2.4] have characterized commutative rings R with $\text{girth}(\Gamma(R)) = 4$. Their findings about this can be summarized in the following theorem.

Theorem 6.1 (Anderson, Mulay, [8]). *If R is a commutative ring with identity such that $\text{girth}(\Gamma(R)) = 4$, then*

- (1) *either $\Gamma(R) = K_{m,n}$, $m, n \geq 2$ and the total quotient ring of R is a direct product of two fields $F_1 \times F_2$, $|F_i| \geq 3$,*
- (2) *or $\Gamma(R) = \overline{K}_{3,m}^m$ and $R = D \times B$, where D is an integral domain with at least 3 elements and $B \in \{\mathbb{Z}_4, \mathbb{Z}_2[X]/(X^2)\}$.*

The following examples show that there exist large families of commutative semirings with their zero-divisor graphs equal to $\overline{K}_{n,m}^r$.

Example 6.2. Let $S = \{0, a_1, a_2, \dots, a_{m-1}, 1\}$ be a totally ordered lattice and $T \subseteq \mathcal{M}_{n-1}(\mathcal{B})$ the commutative semiring, generated by $A = J_{n-1}^{n-2}$ and $B = I_{n-1} + J_{n-1}$. Note that $Z(S) = \{0\}$ and $Z(T) = \{0, A\}$. Then, $Z(S \times T)$ consists of two types of elements $(s, t) \in S \times T$: the first type are those having $s = 0$ or $t = 0$, which form the induced subgraph $K_{n,m}$ of $\Gamma(S \times T)$; the second type are the elements having $t = A$, which are all neighbours of the vertex $(0, A)$. Thus, $\Gamma(S \times T) = \overline{K}_{n,m}^m$. \square

Example 6.3. Let $S = \mathcal{M}_2(\mathcal{B})$ and consider the semiring S^{2n-3} for some $k \in \mathbb{N}$. Denote by e_i the element in S^{2n-3} , which has its only nonzero entry equal to I_2 in the i -th position and moreover let $I = I_2$ and $J = J_2$. Let us define the following elements:

$$\begin{aligned} a &= Je_1 & b &= e_2 \\ c_j &= (I + J)e_1 + \sum_{i=1}^j Je_{2i+1} \text{ for } j = 1, 2, \dots, n-2 & d &= e_2 + e_4 \\ e &= \sum_{i=0}^{n-2} Je_{2i+1} + \sum_{i=1}^{n-2} e_{2i} \end{aligned}$$

Consider the semiring T generated by $\{1, a, b, c_1, c_2, \dots, c_{n-2}, d, e\}$ and observe that $\Gamma(T) = \overline{K}_{n,2}^3$. \square

The following theorem shows that all zero-divisor graphs with their girth equal to 4 are actually of this form.

Theorem 6.4. *If S is a commutative semiring and $\text{girth}(\Gamma(S)) = 4$ then $\Gamma(S) = \overline{K}_{m,n}^r$.*

Proof. Since $\text{girth}(\Gamma(S)) = 4$, $\Gamma(S)$ contains $K_{2,2}$ as induced subgraph. We proceed inductively by adding vertices while always maintaining $\Gamma(S) = \overline{K}_{\mu,\nu}^\rho$ for some ρ, μ, ν .

Assume that we have in $\Gamma(S)$ an induced subgraph $\overline{K}_{\mu,\nu}^\rho$, $\mu, \nu \geq 2$. Let us decompose the vertex set of $\overline{K}_{\mu,\nu}^\rho$ into 3 sets: $V_1 = \{v; \deg v = 1\}$ (possibly empty), V_2 and V_3 are the bipartite parts of $K_{\mu,\nu}$, where each vertex in V_3 has degree μ . Moreover, let $a \in V_2$ be the vertex with $\deg(a) = \nu + \rho$. If $\rho = 0$, then choose a to be any vertex in V_2 .

Choose any vertex $x \in \Gamma(S)$, that is not in $\overline{K}_{\mu,\nu}^\rho$.

- If $x - v$ is an edge for some $v \in V_1$, then by Theorem 3.1, $x - w$ is an edge for some $w \in V_2 \cup V_3$. If $w \in V_2$, then $\Gamma(S)$ contains $C_{4,5}$ as an induced subgraph, and if $w \in V_3$, then $\Gamma(S)$ contains $C_{4,4}$ as an induced subgraph. Both conclusions contradict Lemma 5.3.
- If $\deg(x) = 1$ and $x - a$ is an edge, or $V_1 = \emptyset$, then we get $\overline{K}_{\mu,\nu}^{\rho+1}$.
- If $V_1 \neq \emptyset$ and $\deg(x) = 1$, then if $x - v$ is an edge for some $v \in V_2 \setminus \{a\}$, then $\Gamma(S)$ contains C_4'' as an induced subgraph and otherwise, if $x - v$ is an edge for some $v \in V_3$, then $\Gamma(S)$ contains C_4' as an induced subgraph, which contradicts Lemma 5.2.
- If $\deg(x) \geq 2$ and $w - x - v$ is a path, then $v, w \in V_2$ or $v, w \in V_3$. (Otherwise, $\text{girth}(\Gamma(S)) = 3$.) Say, $v, w \in V_2$. Suppose there exists $u \in V_2$ such that $xu \neq 0$. Now, $(xu)y = (xu)w = (xu)v = 0$ for all $y \in V_3$, and this contradicts the assumption that $\text{girth}(\Gamma(S)) = 4$. Thus, $x - u$ is an edge in $\Gamma(S)$ for all $u \in V_2$, so we get $\overline{K}_{\mu,\nu+1}^\rho$. Similarly, if $v, w \in V_3$, we get $\overline{K}_{\mu+1,\nu}^\rho$. ■

The next observation is a semiring generalization of a result that appears in [6] for the ring theoretic case.

Proposition 6.5. *If S is a commutative semiring with $\text{girth}(\Gamma(S)) = 4$, then all nilpotents are of the order equal to 2.*

Proof. Note that since $\text{girth}(\Gamma(S)) = 4$, graph $\Gamma(S)$ does not contain any triangles.

Suppose $x \in \mathcal{N}(S)$ and $x^n = 0$, $x^{n-1} \neq 0$, $n \geq 3$. Thus, $x - x^{n-1}$ is an edge in $\Gamma(S)$. Note that $\deg(x) = 1$ since otherwise $xy = 0$ implies that $x - y - x^{n-1} - x$ is a triangle in $\Gamma(S)$.

In $\Gamma(S)$ there exists a 4-cycle $a - b - c - d - a$ and since $\text{diam}(\Gamma(S)) \leq 3$, it follows that $x^{n-1} \in \{a, b, c, d\}$. Say, $x^{n-1} = d$. Then $(bx)x^{n-1} = (bx)a = (bx)c = 0$ and $bx \neq 0$. Since $\Gamma(S)$ does not contain any triangles, $bx = x^{n-1}$. Similarly, $bx^{n-1} = x^{n-1}$. Now, $x^{n-1} = bx^{n-1} = bxx^{n-2} = x^{2n-3} = 0$, which is a contradiction. It follows that $n = 2$. ■

7. COMMUTATIVE SEMIRINGS HAVING ZERO-DIVISOR GRAPHS WITH ONE 3-CYCLE

We now proceed to a description of all graphs with their girths equal to 3, with an additional assumption that they contain exactly one 3-cycle.

The main purpose of the last two sections is to obtain the characterization of all rings (or equivalently all additively cancellative semirings) having the zero-divisor graph with one 3-cycle, which is a step towards the characterization of rings with the girth of their zero-divisor graph equal to 3.

Proposition 7.1. *If S is a commutative semiring and $\Gamma(S)$ contains exactly one 3-cycle, then $\Gamma(S) = K_{m,n}^{\Delta(r_1, r_2, r_3)}$.*

Proof. If $\Gamma(S)$ contains an n -cycle for some $n \geq 4$ then it also contains $C_{4,3}$, $a - b - c - d - a - e - d - a$ as an induced subgraph by Lemma 5.1.

We proceed by adding arbitrary vertices from $\Gamma(S)$ to this subgraph, while showing that in the process we always maintain the structure of $\Gamma(S) = \overline{K}_{\mu, \nu}^{\Delta(\rho_1, \rho_2, \rho_3)}$ for some ρ, μ, ν .

Assume that in $\Gamma(S)$, we have an induced subgraph $\overline{K}_{\mu, \nu}^{\Delta(\rho_1, \rho_2, \rho_3)}$, $\mu, \nu \geq 2$. Let us decompose the vertex set of $\overline{K}_{\mu, \nu}^{\Delta(\rho_1, \rho_2, \rho_3)}$ into 4 sets: $V_1 = \{v; \deg v = 1\}$ (possibly empty), V_2 and V_3 are the bipartite parts of $K_{\mu, \nu}$, where each vertex in V_3 has degree μ , and $V_4 = \{e\}$, the top of the 3-cycle.

Choose any vertex $x \in \Gamma(S)$, that is not in $\overline{K}_{\mu, \nu}^{\Delta(\rho_1, \rho_2, \rho_3)}$ and add it to the graph.

- If $x - v$ is an edge for some $v \in V_1$, then by Theorem 3.1, $x - w$ is an edge for some $w \in V_2 \cup V_3$. If $w \in V_2$, then $\Gamma(S)$ contains $C_{4,5}$ as an induced subgraph, and if $w \in V_3$, then $\Gamma(S)$ contains $C_{4,4}$ as an induced subgraph. Both conclusions contradict Lemma 5.3.
- If $\deg(x) = 1$ and x is a neighbour of a , d or e , then we get $\overline{K}_{\mu, \nu}^{\Delta(\rho_1+1, \rho_2, \rho_3)}$, $\overline{K}_{\mu, \nu}^{\Delta(\rho_1, \rho_2+1, \rho_3)}$ or $\overline{K}_{\mu, \nu}^{\Delta(\rho_1, \rho_2, \rho_3+1)}$, respectively.
- If $\deg(x) = 1$ and x is not a neighbour of a , d and e , let us assume without loss of generality that $x - v$ is an edge for some $v \in V_2 \setminus \{a, d\}$. Since $\deg(x) = 1$, then $xa \neq 0$ and $(xa)y = (xa)v = (xa)e = 0$ for all $y \in V_3$, and this contradicts the assumption that $\Gamma(S)$ has exactly one 3-cycle.
- If $\deg(x) \geq 2$ and $w - x - v$ is a path, then $v, w \in V_2$ or $v, w \in V_3$. (Otherwise, we obtain a new 3-cycle in $\Gamma(S)$ if $w \in V_2$ and $v \in V_3$ or if w and v are two vertices of the 3-cycle $a - e - d - a$, and we obtain $C_{4,4}$ if one of v, w is equal to e .) Say, $v, w \in V_2$. Suppose there exists $u \in V_2$ such that $xu \neq 0$. Now, $(xu)y = (xu)w = (xu)v = 0$ for all $y \in V_3$, and this contradicts the assumption that $\Gamma(S)$ has exactly one 3-cycle. Thus, $x - u$ is an edge in $\Gamma(S)$ for all $u \in V_2$, so we get $\overline{K}_{\mu+1, \nu}^{\Delta(\rho_1, \rho_2, \rho_3)}$. Similarly, if $v, w \in V_3$, we get $\overline{K}_{\mu+1, \nu}^{\Delta(\rho_1, \rho_2, \rho_3)}$.

If the only cycle in $\Gamma(S)$ is the 3-cycle, then all other vertices in $\Gamma(S)$ are at distance 1 from the triangle. Otherwise, if $a - b - e - a - x - y$ is a subgraph of $\Gamma(S)$, but then $xb \neq 0$ and $(xb)a = (xb)e = (xb)y = 0$ which is a contradiction, since we obtain a new 3-cycle in $\Gamma(S)$. ■

Corollary 7.2. *If S is a commutative semiring with the only cycle of $\Gamma(S)$ being a 3-cycle, then $\Gamma(S) = K_{1,1}^{\Delta(r_1, r_2, r_3)}$.*

The following example shows that there exist commutative semirings with their zero-divisor graphs equal to $K_{1,1}^{\Delta(r_1, r_2, r_3)}$ for all $r_1, r_2, r_3 \geq 1$.

Example 7.3. *Let us denote by e_i the element in $\mathcal{B}^{r_1+r_2+r_3}$, which has its only nonzero entry in the i -th position and f_j the element in $\mathcal{B}^{r_1+r_2+r_3}$, which has its only zero entry in the j -th position. Let us define elements*

$$\begin{aligned} a_i &= f_i \sum_{t=1}^{r_1-1} e_t + e_{r_1+r_2} + e_{r_1+r_2+r_3} & \text{for all } i = 1, 2, \dots, r_1 - 1, \\ b_j &= f_{r_1+j} \sum_{t=1}^{r_2-1} e_{r_1+t} + e_{r_1} + e_{r_1+r_2+r_3} & \text{for all } j = 1, 2, \dots, r_2 - 1, \\ c_\ell &= f_{r_1+r_2+\ell} \sum_{t=1}^{r_3-1} e_{r_1+r_2+t} + e_{r_1} + e_{r_1+r_2} & \text{for all } \ell = 1, 2, \dots, r_3 - 1. \end{aligned}$$

Denote by S the semiring, generated by

$$\begin{aligned} \mathcal{Z} &= \{e_{r_1}, e_{r_1+r_2}, e_{r_1+r_2+r_3}, a_i, b_j, c_\ell; \\ &= 1, 2, \dots, r_1 - 1, j = 1, 2, \dots, r_2 - 1, \ell = 1, 2, \dots, r_3 - 1\} \end{aligned}$$

and note that $Z(S)^* = \mathcal{Z}$. Clearly,

$$\begin{array}{ll} e_{r_1} - a_i & e_{r_1} - (e_{r_1+r_2} + e_{r_1+r_2+r_3}) \\ e_{r_1+r_2} - b_j & e_{r_1+r_2} - (e_{r_1} + e_{r_1+r_2+r_3}) \\ e_{r_1+r_2+r_3} - c_\ell & e_{r_1+r_2+r_3} - (e_{r_1} + e_{r_1+r_2}) \end{array}$$

are edges in $\Gamma(S)$ for all $i = 1, 2, \dots, r_1 - 1$, $j = 1, 2, \dots, r_2 - 1$ and $\ell = 1, 2, \dots, r_3 - 1$, and $e_{r_1} - e_{r_1+r_2} - e_{r_1+r_2+r_3} - e_{r_1}$ form a 3-cycle. Thus, $\Gamma(S) = K_{1,1}^{\Delta(r_1, r_2, r_3)}$. \square

Recall that we proved in Lemma 5.1 that all zero-divisor graphs containing exactly one 3-cycle and at least one k -cycle, $k \geq 4$, also contain $C_{4,3}$ as an induced subgraph. The following technical lemma will give us some algebraic properties on the elements, corresponding to the vertices of $C_{4,3}$. It will enable us to prove that in this case $\Gamma(S) = K_{m,n}^{\Delta(r_1, r_2, 0)}$ where $r_1, r_2 \geq 1$, $m, n \geq 2$.

Lemma 7.4. *Suppose S is a commutative semiring, $\Gamma(S)$ contains exactly one 3-cycle and at least one k -cycle, $k \geq 4$. Let us denote by $a - b - c - d - a - e - b - a$ its induced subgraph and let $a - f$ be an edge in $\Gamma(S)$ and $\deg f = 1$. Then,*

- (1) $a^2 = b^2 = e^2 = 0$,
- (2) $ac = ec = fc = a$,
- (3) $bd = ed = b$,
- (4) $a + b = e$.

Moreover, if S is additively cancellative, then

- (5) $2a = 2b = 2e = 0$,
- (6) $b + e = a$ and $a + e = b$.

Proof. Firstly, let us note that $ec \neq 0$ and $(ec)a = (ec)b = (ec)d = 0$, and since $\Gamma(S)$ contains only one 3-cycle, we have $ec = a$. Similarly we prove that $ac = fc = a$ and $bd = ed = b$.

Consider now the element a^2 . Since $a^2e = a^2b = 0$, and $a^2 \neq b$ (otherwise $bd = a^2d = 0$), $a^2 \neq e$ (otherwise $be = a^2e = 0$), $a^2 \neq a$ (otherwise $a^2 = aa = aec = 0$), it follows that $a^2 = 0$. Similarly we prove that $b^2 = 0$.

Now, let us observe that $(a+b)a = (a+b)b = (a+b)e = 0$. Note that $a+b \neq 0$ (otherwise, $bd = (a+b)d = 0$), $a+b \neq a$ (otherwise, $bd = (a+b)d = ad = 0$), $a+b \neq b$ (otherwise, $ac = (a+b)c = bc = 0$), and since $\Gamma(S)$ contains only one 3-cycle, we have $a+b = e$ and therefore also $e^2 = a^2 + b^2 + 2ab = 0$.

Suppose now, S is additively cancellative. Then, $2a \neq a$ and $(2a)b = (2a)e = (2a)d = 0$, thus $2a = 0$. Similarly, $2b = 0$ and therefore also $2e = 2(a+b) = 0$. Now, it follows that $a = a+b+b = e+b$ and $b = b+a+a = e+a$. ■

Theorem 7.5. *If S is a commutative semiring and $\Gamma(S)$ contains exactly one 3-cycle and at least one k -cycle, $k \geq 4$, then $\Gamma(S) = K_{m,n}^{\Delta(r_1, r_2, 0)}$ and $r_1, r_2 \geq 1$, $m, n \geq 2$.*

Proof. If $\Gamma(S)$ contains a k -cycle for some $k \geq 4$ then it also contains $C_{4,3}$, $a-b-c-d-a-e-b-a$ as an induced subgraph by Lemma 5.1. By Proposition 7.1, $\Gamma(S) = K_{m,n}^{\Delta(r_1, r_2, r_3)}$ and let $a_i, b_j, e_\ell \in S$ such that $\deg(a_i) = \deg(b_j) = \deg(e_\ell) = 1$ and $a-a_i, b-b_j, e-e_\ell$ are edges in $\Gamma(S)$ for $i = 1, 2, \dots, r_1$, $j = 1, 2, \dots, r_2$ and $\ell = 1, 2, \dots, r_3$.

Now, let us prove that $r_1 \geq 1$. Consider the sum $d+e$ and observe that $d+e \neq 0$. Namely, if $d+e = 0$, we would have $ec = (e+d)c = 0$. Since $(e+d)a = 0$, we have few possibilities for $e+d$. Because $e+d \neq a$ (otherwise $db = (d+e)b = ab = 0$), $e+d \neq b$ (otherwise $db = (d+e)b = b^2 = 0$ by Lemma 7.4 (1)), $e+d \neq e$ (otherwise $db = (d+e)b = eb = 0$) and $c(d+e) \neq 0$ (otherwise $a = ce = c(d+e) = 0$), it follows that $d+e = a_i$ for some i . Similarly, we prove that $e+c = b_j$ for some j and therefore $r_1, r_2 \geq 1$.

Assume $r_3 \geq 1$ and consider an element $ae_\ell \neq 0$. We have $(ae_\ell)b = (ae_\ell)e = (ae_\ell)d = (ae_\ell)a_i = 0$ and since $\Gamma(S)$ contains only one 3-cycle, we have $ae_\ell = a$. Similarly we prove that $be_\ell = b$. Now, by Lemma 7.4 (4) we have that $e = a+b = ae_\ell + be_\ell = (a+b)e_\ell = ee_\ell = 0$, which is a contradiction. Thus, $r_3 = 0$ and $\Gamma(S) = K_{m,n}^{\Delta(r_1, r_2, 0)}$. ■

Corollary 7.6. *If S is an additively cancellative commutative semiring and $\Gamma(S)$ contains exactly one 3-cycle and at least one k -cycle, $k \geq 4$, then $\Gamma(S) = K_{m,n}^{\Delta(n-1, m-1, 0)}$ with $m, n \geq 2$.*

Proof. By Theorem 7.5 we know that $m, n \geq 2$, and denote (as in the proof of the same theorem) by $a-b-c_j-d_i-a-e-b-a$ the induced subgraph of $\Gamma(S)$, let $d_1, d_2, \dots, d_{n-1}, b$ and $c_1, c_2, \dots, c_{m-1}, a$ be the partition of vertices of an induced complete bipartite subgraph of $\Gamma(S)$. By Lemma 7.4 (1) we have that $e^2 = 0$ and therefore $(d_i+e)a = 0$, $(d_i+e)e \neq 0$, $(d_i+e)b \neq 0$ and $(d_i+e)c_j \neq 0$ for all $i = 1, 2, \dots, n-1$. Therefore, $\deg(d_i+e) = 1$ and $(d_i+e)-a$ is an edge in $\Gamma(S)$. Since $d_i+e \neq d_j+e$ for $i \neq j$, it follows that $r_1 \geq n-1$. Similarly, we can see $r_2 \geq m-1$.

Let $a-f$ be an edge in $\Gamma(S)$ and $\deg f = 1$. Using Lemma 7.4, observe that $(f+a)a = 0$, $(f+a)c_i = fc + ac_i = a + a = 0$ and $(f+a)e \neq 0$ and thus $f+a = d_j$ for some j . Now, $d_j + b = f + a + b = f + e$ and since $(d_j + b)a = (d_j + b)c_i = 0$ for all i and S is additively cancellative, it follows that $d_k = d_j + b = f + e$. By adding e it follows that $d_k + e = f$, which proves that $r_1 = n - 1$. Similarly, $r_2 = m - 1$. ■

Every additively cancellative semiring can be embedded into a ring of differences (see for example [20, Thm. 5. 11]), but in case the zero-divisor graph of the additively cancellative semiring contains exactly one 3-cycle, we can prove that the semiring actually has to be a ring. We will then study the zero-divisor graphs of rings in the next section.

Proposition 7.7. *If S is a commutative additively cancellative semiring and $\Gamma(S)$ contains exactly one 3-cycle, then S is a ring.*

Proof. Denote the only 3-cycle in $\Gamma(S)$ by $a - b - e - a$. Now $(2a)b = (2a)e = 0$, so $2a \in Z(S)$ and $2a \in \{0, b, e\}$, since $2a = a$ implies that $a = 0$. Similarly, we can show that $3a \in \{0, b, e\}$ and $4a \in \{0, b, e\}$. So, either at least one of $2a, 3a, 4a$ is equal to zero or at least two of $2a, 3a, 4a$ coincide. Since S is additively cancellative, it follows that in all cases there exists an integer $n > 0$ such that $na = 0$. Similarly, we can also show that $mb = re = 0$ for some integers $m, r > 0$. This implies that $(nm)a = (mn)b = 0$, so $nm \in Z(S)$ and $nm \in \{0, a, b, e\}$. In each case we get that $N = 0$ for some integer $N > 0$, so for every $x \in S$ we have $-x = (N - 1)x \in S$, therefore S is a ring. ■

The following example shows that in case S is not additively cancellative, the zero-divisor graph $\Gamma(S) = K_{m,n}^{\Delta(r_1, r_2, 0)}$ need not have $r_1 = n - 1$ and $r_2 = m - 1$.

Example 7.8. *Let $S \subseteq \mathcal{M}_2(\mathcal{B}) \times \mathcal{M}_2(\mathcal{B}) \times \mathcal{M}_2(\mathcal{B})$ be a commutative semiring, generated by*

$$\begin{array}{lll} a = (J_2, 0, 0) & b = (0, 0, J_2) & c = (I_2 + J_2, 0, 0) \\ d = (0, 0, I_2 + J_2) & f = (J_2, J_2, I_2 + J_2) & 1 = (I_2, I_2, I_2) \end{array}$$

Observe that $Z(S)^ = \{a, b, c, d, a + b, f, a + d, b + c\}$ and that $\Gamma(S) = K_{2,2}^{\Delta(2,1,0)}$.* □

8. COMMUTATIVE RINGS HAVING ZERO-DIVISOR GRAPHS WITH ONE 3-CYCLE

We now know the types of graphs that can appear as the zero-divisor graphs of semirings. However, the setting appears to be too general to allow for a classification of the structure of semirings that have these types of zero divisor graphs. We will characterize all commutative rings (with identity), such that their zero divisor graphs contain exactly one 3-cycle.

For an arbitrary ring R , let $T_2(R)$ denote the ring of all matrices of the form $aI_2 + bJ_2$, where $a, b \in R$.

Lemma 8.1. *If $S = S_1 \times S_2$ is an additively cancellative commutative semiring such that $\Gamma(S)$ contains exactly one 3-cycle, then $S_1, S_2 \in \{\mathbb{Z}_4, T_2(\mathbb{Z}_2)\}$ are rings and $\Gamma(S) = K_{3,3}^{\Delta(2,2,0)}$.*

Proof. Suppose firstly that all 3 vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ on the 3-cycle have one component equal to 0. Since $\Gamma(S)$ contains exactly one 3-cycle, it follows that at least one of x_1, x_2, x_3 is nonzero and at least one of y_1, y_2, y_3 is nonzero. So, let us assume that $(a, 0) - (b, 0) - (0, c) - (a, 0)$ is a 3-cycle. If $a^2 = 0$, then $(a, 0) - (a, c) - (b, 0) - (a, 0)$ is another 3-cycle in the graph and if $a^2 \neq 0$, then $(b, 0) - (a^2, 0) - (0, c) - (b, 0)$ is another 3-cycle in the graph, contradiction.

Let $(a, b), (a_2, b_2)$ and (a_3, b_3) be the vertices on the 3-cycle and suppose that $a, b \neq 0$. Since $(a, 0)$ and $(0, b)$ are also zero divisors, we have $\{(a_2, b_2), (a_3, b_3)\} = \{(a, 0), (0, b)\}$ and $a^2 = b^2 = 0$. If there exists $x \in Z(S_1)$, $x \notin \{0, a\}$, then $xy = 0$ for some $y \in Z(S_1)^*$, so either $(x, 0) - (y, 0) - (0, b) - (x, 0)$ is another 3-cycle (if $x \neq y$) or $(x, 0) - (x, b) - (0, b) - (x, 0)$ is another 3-cycle (if $x = y$). Thus, $Z(S_1) = \{0, a\}$ and $Z(S_2) = \{0, b\}$. Since $2a \in Z(S_1)$ and S_1 is additively cancellative, it follows that $2a = 0$. Now, choose an $x \in S_1 \setminus Z(S_1)$. Note that $xa \in Z(S_1)$ implies $xa = a$. Since $(2x)a = 0$, it follows that $2x = 0$ or $2x = a$. Also, $(x + 1)a = 0$, so either $x + 1 = 0$ or $x + 1 = a$. By adding either x or $x + a$ to these equations, we can conclude that $x = 1$ or $x = 1 + a$. Thus we proved that $S_1 = \{0, 1, a, 1 + a\}$. Since $a^2 = 2a = 0$ and we either have $1 + 1 = 0$ or $1 + 1 = a$, it follows that either $S_1 \simeq T_2(\mathbb{Z}_2)$ (via mapping $1 \mapsto I_2, a \mapsto E_{1,2}$) or $S_1 \simeq \mathbb{Z}_4$. Similarly, we show that $S_2 \in \{\mathbb{Z}_4, T_2(\mathbb{Z}_2)\}$. ■

In the case S is a ring, the following Proposition shows that the assumption that S is a direct product is actually superfluous.

Proposition 8.2. *Let R be a commutative ring with identity such that $\Gamma(R)$ contains exactly one 3-cycle and at least one k -cycle, $k \geq 4$, then R is isomorphic to a direct product $R_1 \times R_2$, where $R_1, R_2 \in \{\mathbb{Z}_4, T_2(\mathbb{Z}_2)\}$ and $\Gamma(R) = K_{3,3}^{\Delta(2,2,0)}$.*

Proof. By Corollary 7.6 it follows that $\Gamma(R) = K_{m,n}^{\Delta(n-1,m-1,0)}$ with $m, n \geq 2$. Denote by $a - b - c_i - d_j - a - e - b - a$ the induced subgraph of $\Gamma(R)$ where $d_1, d_2, \dots, d_{n-1}, b$ and $c_1, c_2, \dots, c_{m-1}, a$ is the partition of vertices of an induced complete bipartite subgraph of $\Gamma(R)$.

By Lemma 7.4 we know that $ec_i = a$ and therefore $(c_i - c_j)e = (c_i - c_j)b = (c_i - c_j)d_k = 0$ for all i, j, k . Thus $c_i - c_j = a$ for all $i \neq j$ and $m, n \leq 3$.

Observe that $(b + c_i)b = 0$ and $c_i^2 \neq 0$, because $c_i^2 = 0$ yields $(b + c_i)c_i = 0$ and since $b + c_i \neq 0$, $b + c_i \neq b$, $b + c_i \neq c_i$ and $b + c_i \neq d_j$ (otherwise, $ac_i = a(b + c_i) = ad_j = 0$), we get a contradiction. Since $c_i c_j = a$ together with Lemma 7.4 implies that $c_i a = c_i c_j e = ae = 0$, it follows that $c_i c_j = c_k$ for some k . If $m = 2$, then $c_1^2 = c_1$. Suppose that $m = 3$ and assume without loss of generality that $c_1 c_2 = c_1$. If c_2 is not idempotent, i.e., $c_2^2 = c_1$, it follows that $c_1^2 = c_1 c_2^2 = c_1 c_2 = c_1$ and therefore c_1 is idempotent. We proved that R in all cases contains an idempotent c . Then, $1 = c + (1 - c)$ is an orthogonal decomposition of identity, thus $R \simeq Rc \times R(1 - c)$. Now, the proposition follows by Lemma 8.1. ■

It remains for us to investigate the zero-divisor graphs with girth equal to 3, containing exactly one cycle.

Lemma 8.3. *If R is a commutative ring with identity and $\Gamma(R) = K_3$, then R is isomorphic to one of the following rings: $T_2(GF(4))$, $\mathbb{Z}_4[x]/(x^2 + x + 1)$, $\mathbb{Z}_2[x, y]/(x^2, xy, y^2)$, or $\mathbb{Z}_4[x]/(2x, x^2)$.*

Proof. If $\Gamma(R) = K_3$, then let $Z(R) = \{0, a, b, e\}$. Suppose there exists $f \in R$ such that $fa = b$ or $fa = e$. Without loss of generality, we can assume that $fa = b$. Then $(f + 1)a = b + a$ and $a + b \in Z(R)$. If $a + b = e$, then $aR = Z(R)$. Otherwise, since R is a ring, $a + b \notin \{a, b\}$, so $a + b = 0$. Then $b = -a$, and $(1 - f)a = a - b = 2a \in Z(R)$. Note that $2a = 0$ implies that $b = -a = a$ and $2a = a$ implies that $a = 0$. So, consider the case $2a = b = -a$. Since $a + e \in Z(R)$ and is obviously not equal to $0, a, e$, we have $a + e = b = -a$ and thus $e = -2a = a$, a contradiction. Therefore, $2a = e$ and again $aR = Z(R)$.

Since R -module Ra is isomorphic to $R/\text{Ann}(a)$, and $\text{Ann}(a) = Z(R)$, we have that $|R| = 16$. By [23, Thm. 12] it follows that $R \simeq T_2(GF(4))$, $R \simeq \mathbb{Z}_4[x]/(x^2 + x + 1)$.

In the remaining case we have that $fa = a$ therefore $(1 - f)a = 0$ for all $f \in R \setminus Z(R)$. We thus have $1 - f \in Z(R)$ and therefore $R = \{0, 1, a, b, e, 1 + a, 1 + b, 1 + e\}$. Since the set of zero divisors is closed under addition, R is a local ring of order 8. By [15, p. 687], R is one of the following:

- $GF(8)$, which has no nontrivial zero-divisors,
- $\mathbb{Z}_2[x]/(x^3)$, but $\Gamma(R) = P_3$,
- $\mathbb{Z}_2[x, y]/(x^2, xy, y^2)$, which gives us $\Gamma(R) = K_3$,
- $\mathbb{Z}_4[x]/(2x, x^2 - 2)$, but $\Gamma(R) = P_3$,
- $\mathbb{Z}_4[x]/(2x, x^2)$, which gives us $\Gamma(R) = K_3$,
- \mathbb{Z}_8 , but $\Gamma(R) = P_3$.

Thus the Lemma follows. ■

We are now in a position to characterize all rings such that their zero-divisor graphs contain exactly one 3-cycle.

Theorem 8.4. *If R is a commutative ring with identity and $\Gamma(R)$ contains exactly one 3-cycle, then exactly one of the following statements holds:*

- (1) R is isomorphic to a direct product $R_1 \times R_2$, where $R_1, R_2 \in \{\mathbb{Z}_4, T_2(\mathbb{Z}_2)\}$ and $\Gamma(R) = K_{3,3}^{\Delta(2,2,0)}$,
- (2) $R \simeq T_2(GF(4))$, $R \simeq \mathbb{Z}_4[x]/(x^2 + x + 1)$, $R \simeq \mathbb{Z}_2[x, y]/(x^2, xy, y^2)$, or $R \simeq \mathbb{Z}_4[x]/(2x, x^2)$, and $\Gamma(R) = K_3 = K_{1,1}^{\Delta(0,0,0)}$,
- (3) $R \simeq \mathbb{Z}_{16}$ or $R \simeq \mathbb{Z}_2[x]/(x^4)$ and $\Gamma(R) = K_{1,1}^{\Delta(4,0,0)}$.

Proof. If $\Gamma(R)$ apart from the 3-cycle also contains an n -cycle for some $n \geq 4$, then Proposition 8.2 implies (1) and if $\Gamma(R) = K_3$, then Lemma 8.3 implies (2).

By Proposition 7.1 the only remaining case is $\Gamma(R) = K_{1,1}^{\Delta(r_1, r_2, r_3)}$, and let $a - b - e - a$ denote the 3-cycle and let $a_i, b_j, e_\ell \in S$ such that $\deg(a_i) = \deg(b_j) = \deg(e_\ell) = 1$ and

$a - a_i, b - b_j, e - e_\ell$ are edges in $\Gamma(S)$ for $i = 1, 2, \dots, r_1, j = 1, 2, \dots, r_2$ and $\ell = 1, 2, \dots, r_3$. Note that $((1 + b_j)a)b = ((1 + b_j)a)e = ((1 + b_j)a)a_i = 0$. This yields $(1 + b_j)a = 0$, since $(1 + b_j)a = a$ implies that $b_j a = 0$.

If $r_1, r_2, r_3 > 0$ then note that $b_j e = e$ and then $(1 + b_j)e = e + e = 2e = 0$, since $(2e)a = (2e)b = (2e)e_\ell = 0$ and $2e \neq e$. Since $(1 + b_j)b \neq 0$, we have $1 + b_j = b$. By multiplying this equation with b , we obtain $b^2 = b$ and this gives us a contradiction by Lemma 8.1. Therefore $r_3 = 0$.

Now, we shall prove that also $r_2 = 0$. Since the left R -module Re is isomorphic to the quotient module $R/Ann(e)$, and both Re and $Ann(e)$ have at most 4 elements $(0, e, a, b)$, we know that R is a ring of at most 16 elements. We also know that R is a directly indecomposable ring by Lemma 8.1, therefore it contains no non-trivial idempotents. Thus, by [22, Theorem VII.7] R is a local ring and the set of zero divisors $Z(R)$ is the Jacobson radical of R . Assume that $r_2 > 0$. Similarly as above, we can see that $(1 + b_j)a \neq a$, so $(1 + b_j)a = 0$ and thus $1 + b_j$ is a zero divisor. Since b_j is a zero divisor as well, we have that $1 = 1 + b_j + (-b_j)$ is a zero divisor, which gives us a contradiction.

Therefore, $\Gamma(R) = K_{1,1}^{\Delta(r_1, 0, 0)}$ and we can assume that $r_1 > 0$. Note that $(a^2)a_i = (a^2)b = (a^2)e = 0$, so $a^2 = 0$ since there are no non-trivial idempotents in R . Then for each a_i such that $a_i a = 0$ we also obtain $(a_i + a)a = (a_i + b)a = (a_i + e)a = 0$. Observe that $a_i + a, a_i + b, a_i + e \notin \{0, a, e, b\}$: for example, if $a_i + b = a$ then $a_i e = 0$; if $a_i + b = e$ then since $(a + b)a = (a + b)e = 0$, it follows that $a + b = e$, and therefore $a_i = a$. Similarly, we treat other cases.

Therefore $r_1 \geq 4$ and thus $|Z(R)| \geq 8$. Since $Z(R) \neq R$ this implies that $|R| = 16$. Now, all rings of order 16 are listed in [15, pages 687–690], and we can check which ones are commutative rings such that their zero-divisor graph only has one 3-cycle. Among the rings of characteristic 2, the only suitable ring is $\mathbb{Z}_2[x]/(x^4)$, since all 3 commutative rings in [15, page 687, case 1.2] have $J^2 = \{0, a\}$ and the generators x_1, x_2, a of J give us an additional 3-cycle, either $a - x_1 - x_2 - a$ in case $x_1 x_2 = 0$, or $a - a_i - a + a_i - a$ in case $x_1^2 = x_2^2 = 0$. Similarly, we can deal with the [15, page 689, case 2.2.a], which proves that there are no such rings of characteristic 4. It is easy to check all the remaining cases to see that the only other ring that can occur is the ring \mathbb{Z}_{16} . ■

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